# A spacetime with closed timelike geodesics everywhere

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**Abstract** In the present article we find a new class of solutions of Einstein's field equations. It describes stationary, cylindrically symmetric spacetimes with closed timelike geodesics everywhere outside the symmetry axis. These spacetimes contain a magnetic field parallel to the axis, a perfect fluid with constant density and pressure, and Lorentz invariant vacuum with energy density represented by a negative cosmological constant.

## 1. Introduction

Only three known solutions of Einstein's field equations representing spacetimes in $\mathbf{R}^4$ with closed timelike geodesics have previously been found. It was demonstrated by Steadman [1] that such curves exist in the exterior van Stockum spacetime [2] representing spacetime inside and outside a cylindrically symmetric rotating dust distribution. It has also been shown by Bonnor and Steadman [3] that a spacetime with two spinning particles, each one with a magnetic moment and mass equal to their charge, permit special cases in which there exist closed timelike geodesics. More recently Grøn and Johannesen [4] found a class of solutions of Einstein's field equations with closed timelike geodesics representing spacetime outside a spinning cosmic string surrounded by a region of finite radial extension with vacuum energy and a gas of non-spinning strings. In all of these spacetimes there are closed timelike geodesics only at particular radii.

In the present article we present a solution of Einstein's field equations representing a cylindrically symmetric spacetime with closed timelike geodesics everywhere outside the axis.

## 2. A spacetime with closed timelike geodesics everywhere

We shall here investigate a class of stationary cylindrically symmetric spacetimes described by a line element of the form

$$ds^2 = -(dt - 2\,\omega\, a(r)\, d\phi)^2 + b(r)^2 d\phi^2 + dr^2 + dz^2 \;, \tag{1}$$

where $\omega$ is a constant, using units so that $c = 1$. The coordinate time is shown on standard clocks at rest in the coordinate system. The presence of the product term $4\,\omega\, a(r)\, dt d\phi$ in the line element means that the coordinate clocks are not Einstein synchronized. Furthermore we assume that there is no gap in the coordinate time along a closed curve around

the axis.

With this form of the line element, Weyssenhoff's formula for the vorticity gives [5]

$$\Omega = \frac{a'}{b}\,\omega \ , \tag{2}$$

where $a'$ is the derivative of $a$ with respect to $r$. We shall consider a space with constant vorticity, $\Omega = \omega$. Hence $b = a'$.

In the present paper we shall search for solutions of Einstein's field equations with closed timelike geodesics at all radii $r > 0$.

Let us consider circular timelike curves in the plane $z =$ constant with center on the z-axis. For such curves to be closed in spacetime, the condition $a'^2 - 4\omega^2 a^2 < 0$, must be fullfilled [4]. Closed timelike geodesics of this type exist in a region where in addition $a'(a'' - 4\omega^2 a) = 0$. We have two cases:
1. $a' = 0$ which is not permitted because it implies that the determinant of the metric tensor vanishes.
2. $a' \neq 0$ and $a'' - 4\omega^2 a = 0$ which gives that

$$a = C_1 \cosh(2\omega r) + C_2 \sinh(2\omega r) \ . \tag{3}$$

From the conditions above it follows that the constants $C_1$ and $C_2$ must fullfill $C_1 > C_2 \geq 0$, where we have assumed that $a > 0$, which involves no loss of physical generality. Since these conditions may be fullfilled everywhere outside the z-axis, we can conclude that there are closed timelike geodesics of the type described above in the entire spacetime outside the z-axis.

Einstein's field equations now give for the mixed components of the total energy momentum tensor

$$\kappa T^{\mu}{}_{\nu} = \omega^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} + \Lambda \delta^{\mu}{}_{\nu} \ , \tag{4}$$

where $\kappa$ is Einstein's gravitational constant, and $\Lambda$ is the cosmological constant representing Lorentz invariant vacuum energy (LIVE).

## 3. The physical contents of this spacetime

In addition to LIVE the spacetime is filled with a perfect fluid and a cylindrically symmetric magnetic field along the z-axis.

The energy momentum tensor of the perfect fluid has components

$$T^{\mu}{}_{\nu} = (\rho + p) u^{\mu} u_{\nu} + p \delta^{\mu}{}_{\nu} \ , \tag{5}$$

where $\delta^{\mu}{}_{\nu}$ is the Kronecker symbol. The fluid is at rest in the coordinate system. Then the contravariant components of the fluid's 4-velocity are $u^t = 1$, $u^i = 0$. The corresponding non-zero covariant components are $u_t = -1$, $u_\phi = 2\omega a$. The non-zero mixed components of the energy momentum tensor are

$$T^t{}_t = -\rho \ , \ \ T^r{}_r = T^{\phi}{}_{\phi} = T^z{}_z = p \ , \ \ T^t{}_{\phi} = 2\omega a(\rho + p) \ . \tag{6}$$

The mixed components of the energy momentum tensor for the magnetic field are

$$T^{\mu}{}_{\nu} = F^{\mu\alpha}F_{\alpha\nu} - \frac{1}{4}\delta^{\mu}{}_{\nu}F_{\alpha\beta}F^{\alpha\beta} \ . \tag{7}$$

The non-vanishing covariant components of the field tensor are

$$F_{r\phi} = -F_{\phi r} = B \ , \tag{8}$$

where $B$ is the magnetic field strength. This gives the following non-vanishing mixed components of the energy momentum tensor

$$T^{t}{}_{t} = -T^{r}{}_{r} = -T^{\phi}{}_{\phi} = T^{z}{}_{z} = \frac{B^2}{2a'^2} \ , \ T^{t}{}_{\phi} = -\frac{2\omega a}{a'^2}B^2 \ . \tag{9}$$

We then obtain the following independent field equations

$$\omega^2 + \Lambda = \kappa\left(\frac{B^2}{2a'^2} - \rho\right) \ , \tag{10}$$

$$\omega^2 + \Lambda = \kappa\left(-\frac{B^2}{2a'^2} + p\right) \ , \tag{11}$$

$$3\omega^2 + \Lambda = \kappa\left(\frac{B^2}{2a'^2} + p\right) \ . \tag{12}$$

Combining equations (10) and (11) we obtain

$$B^2 = (\rho + p)a'^2 \tag{13}$$

in accordance with the field equation for $T^{t}{}_{\phi}$, which hence follows from the other field equations. Subtracting equation (11) from (12) and assuming that the magnetic field points in the positive z-direction, we get

$$B = \sqrt{\frac{2}{\kappa}}\,\omega a' \ . \tag{14}$$

The last two equations give

$$\kappa(\rho + p) = 2\omega^2 \ , \tag{15}$$

which is the equation of state of the perfect fluid. Substituting equation (14) into equation (10) leads to

$$\kappa\rho = -\Lambda = -\kappa\rho_v \ , \tag{16}$$

where $\rho_v$ is the density of the vacuum energy. Equations (15) and (16) show that the density and pressure of the perfect fluid are constant, and that the cosmological constant must be negative in order that the density of the perfect fluid shall be positive. The total energy density of the perfect fluid and the vacuum energy vanishes. The energy density in the spacetime comes from the magnetic field.

## 4. A simple special case

### 4.1. Kinematics

As a simple illustrating example we will consider a solution of the field equations with $C_1 = 1$ and $C_2 = 0$ in equation (3), giving

$$a = \cosh(2\omega r) \ . \tag{17}$$

The line element then takes the form

$$ds^2 = -dt^2 + 4\,\omega\ \cosh(2\omega r)\, d\phi dt - 4\omega^2 d\phi^2 + dr^2 + dz^2 \ , \tag{18}$$

In spite of the minus sign in front of $d\phi^2$ the signature is correct. This is seen from the form (1) of the line element corresponding to the orthogonal basis $(\mathbf{e}_t, \mathbf{e}_\phi + 2\omega a(r)\mathbf{e}_t, \mathbf{e}_r, \mathbf{e}_z)$, since $\mathbf{e}_t \cdot \mathbf{e}_t < 0$, $(\mathbf{e}_\phi + 2\omega a(r)\mathbf{e}_t) \cdot (\mathbf{e}_\phi + 2\omega a(r)\mathbf{e}_t) = b(r)^2 > 0$, $\mathbf{e}_r \cdot \mathbf{e}_r > 0$ and $\mathbf{e}_z \cdot \mathbf{e}_z > 0$.

The 3-space defined by the simultaneity $t =$ constant is given by the spatial line element

$$dl^2 = -4\omega^2 d\phi^2 + dr^2 + dz^2 \ . \tag{19}$$

This shows that the vector $\mathbf{e}_\phi$, which is a tangent vector in this space, has $\mathbf{e}_\phi \cdot \mathbf{e}_\phi = -4\omega^2 < 0$. Hence this vector is timelike.

In the present spacetime the equation of circular null curves in the plane $z =$ constant with center on the z-axis is

$$4\omega^2 \left(\frac{d\phi}{dt}\right)^2 - 4\omega \cosh(2\omega r)\frac{d\phi}{dt} + 1 = 0 \ . \tag{20}$$

The physical velocities of light moving in opposite $\phi$-directions are

$$v_\pm = \sqrt{|g_{\phi\phi}|}\left(\frac{d\phi}{dt}\right)_\pm = 2\omega\left(\frac{d\phi}{dt}\right)_\pm = e^{\pm 2\omega r} \ . \tag{21}$$

The speed of light is diffent from 1 since the coordinate clocks are not Einstein synchronized. These expressions give the intersections of light cones with the plane in the tangent space spanned by $\mathbf{e}_\phi$ and $\mathbf{e}_t$ as shown in Figure 1. The reason for the direction of the light cone is the following. We have considered light signals submitted in the positive and the negative $\phi$-direction. Their angular velocities are denoted by plus and minus in equation (21). One signal has $d\phi > 0$ and the other $d\phi < 0$, but the angular velocities of both have the same sign. Hence the coordinate time interval $dt > 0$ for the first signal, and $dt < 0$ for the second one. Also $|(d\phi/dt)_+| > |(d\phi/dt)_-|$.

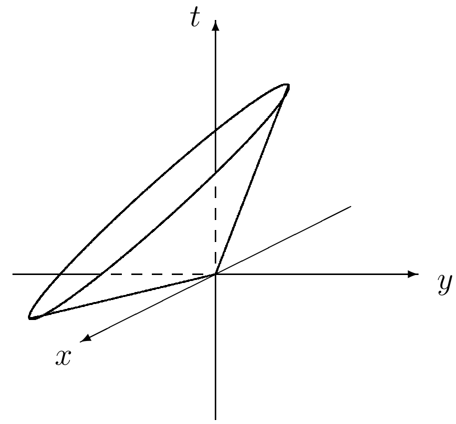


**Figure 1.** Light cones in the spacetime with line element (18) where the x-, y- and t-axes correspond to the basis vectors $\mathbf{e}_r$, $\mathbf{e}_\phi$ and $\mathbf{e}_t$.

This figure shows that the tangent vector to a curve in the $\phi$ direction with $t =$ constant is inside the light cone at each point. Hence such a curve is timelike.

The light signal travelling in the negative $\phi$-direction actually travels backwards in time, and hence can be used to warn people living in previous times about possible catastrophic events in their future. This also involves the possibility of causal paradoxes, and may demand some sort of chronology protection [6].

Let us calculate how far backwards in time light can come by travelling one time around the axis. Integrating equation (21) we find

$$\Delta t_1 = -4\pi\omega e^{-2\omega r_1} \tag{22}$$

for light travelling in the negative $\phi$-direction. Hence by travelling an arbitrary number of times around the axis the light may arrive arbitrarily far backwards in time.

We now consider light moving in both the $\phi$- and $r$-directions. Then the 4-velocity identity for the light takes the form

$$-\dot{t}^2 + 4\,\omega\,\cosh(2\omega r)\,\dot{\phi}\,\dot{t} - 4\omega^2\dot{\phi}^2 + \dot{r}^2 = 0\ , \tag{23}$$

where the dot denotes differentiation with respect to an invariant parameter. This equation shows that there exist null curves with $\dot{t} = 0$ given by

$$4\omega^2\dot{\phi}^2 = \dot{r}^2\ . \tag{24}$$

Integrating this equation, we obtain

$$r = r_0 \pm 2\omega\phi \tag{25}$$

describing Archimedean spirals. Light signals moving along these curves travel infinitely fast, i.e. they arrive at the same point of time $t$ as they are emitted.

### 4.2. Geodesics

The null curves that we have considered are not in general geodesics. We shall now consider timelike and null geodesics. The Lagrangian function of a materal particle or a photon moving in a plane $z =$ constant in a spacetime described by a line element (18) is

$$L = -\frac{1}{2}\dot{t}^2 + 2\,\omega\,\cosh(2\omega r)\,\dot{\phi}\,\dot{t} - 2\omega^2\dot{\phi}^2 + \frac{1}{2}\dot{r}^2\ . \tag{26}$$

Then the $\phi$-component of the geodesic equation is $\ddot{\phi} = 0$, implying that $\dot{\phi} =$ constant along the path. The radial component of the geodesic equation is

$$\ddot{r} = 2\omega^2 \sinh(2\omega r)\,\dot{\phi}\,\dot{t}\ . \tag{27}$$

Hence circular geodesics with $\ddot{r} = 0$ must have $\dot{t} = 0$, and are therefore closed in spacetime. This is the case for both timelike and null geodesics. Thus the null curves going backwards in time, which were considered above, are not geodesics. These photons must move for instance in fibre optic cables.

Considering a material particle moving along a circle in the $z =$ constant plane with center on the z-axis, the 4-velocity identity takes the form

$$-\,\dot{t}^2 + 4\,\omega\,\cosh(2\omega r)\,\dot{\phi}\,\dot{t} - 4\omega^2\dot{\phi}^2 = -1\ . \tag{28}$$

If this circle is closed in spacetime, $\dot{t} = 0$, the particle must have

$$(\dot{\phi})_{\pm} = \pm\frac{1}{2\omega}\ . \tag{29}$$

It can be shown that the geodesic equation is fullfilled for particles moving in this way. Hence particles following closed timelike geodesics in spacetime move with constant angular velocity.

Since $\phi$ is a cyclic coordinate $p_{\phi} = l\dot{\phi} + k\dot{t} =$ constant, where $k = 2\omega a$ and $l = a'^2 - 4\omega^2 a^2$. For $\dot{t} = 0$ this gives $l\dot{\phi} =$ constant. For circular motion the 4-velocity identity of a material particle reduces to $-\dot{t}^2 + 2k\dot{t}\dot{\phi} + l\dot{\phi}^2 = -1$. In order that the circle shall be closed in spacetime, the condition $\dot{t} = 0$ must be fullfilled, which gives $l\dot{\phi}^2 = -1$. Hence $\dot{\phi} =$ constant along the closed timelike geodesic. In general this constant has an $r$-dependent value, permitting $l$ to be a function of $r$ as in [4].

The fact that the angular velocity is constant is a consequence of the axial symmetry, and is valid for arbitrary functions $a(r)$.

## 5. Conclusion

We have constructed solutions of Einstein's field equations in a cylindrically symmetric space with closed timelike geodesics in the entire space outside the symmetry axis. There exist circular geodesics in the plane $z =$ constant with center on the z-axis so that a particle moving freely along one of these geodesics arrives at the same event as it departed from. In addition there exist such timelike and null circular worldlines of material particles or photons going backwards in time. However, these worldlines are not geodesics.

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